A Contribution to the Theory of Magnetoresistance

Wolfgang Kroll

Sommerfeld and Bartlett have shown that the elementary theory of conduction which assumes the existence of a mean free path gives a non-zero longitudinal magnetoresistance, when the quantization of the electronic pathes in the magnetic field is taken into account. In their computation they obtained a quadratic dependence of the change of resistance $\triangle\rho=\rho-\rho_0$ with the magnetic field H. Their result is

$$\frac{\triangle\rho}{\rho}=\left(\frac{1}{8}-\frac{\Lambda}{12}\right)\left(\frac{\mu H}{\varsigma}\right)^2,$$

$\mu=\frac{\lambda h}{2mc}$ the Bohr magneton and ζ the Ferni-energy. Λ is a constant.

Now it is interesting that one can obtain also the observed periodic variation of the magnetoresistance from their formulas, if one computes the sum appearing in the expression not by the Euler-sum formula but by the Poisson-sum formula.

The energylevels of the electrons the magnetic field H, which we assume in the Z-direction are

$$E=(2n+1)\mu H+\frac{p_z^2}{2m}$$

where p_z is the momentum parallel to the field. Denoting by f_0 the undisturbed Fermidistribution

$$f_0=\frac{1}{1+e^{\frac{E-\varsigma}{kT}}}=kT\frac{\partial}{\partial\zeta}\lambda n(1+e^{-\frac{E-\varsigma}{kT}})$$

and by v the velocity of the electron, Sommerfeld and Bartlett write the electric current I in the electric field F parallel to the magnetic field

$$I=C\sum_{n=0}^{\infty}\int_0^{\infty}\left(\frac{p_z}{m}\right)^2\frac{1}{v}\frac{\partial f_0}{\partial E}dp_z \quad \text{with } C=-\frac{16\pi e^2 m\mu HFe\lambda}{h^3}$$

Here l is the mean free path.

Thus we can write with $v=\left(\frac{2E}{m}\right)^{\frac{1}{2}}$

$$I=C\sum_{n=0}^{\infty}\int_0^{\infty}\sqrt{\frac{2E}{m}}\left[1-\frac{2\mu H}{E}\left(n+\frac{1}{2}\right)\right]\frac{\partial f_0}{\partial E}d\rho_z$$

Here we introduce E as a new integration variable and obtain.

$$I=C\sum_{n=o}^{\infty}\int_{2\mu H(n+\frac{1}{2})}^{\infty}[1-\frac{2\mu H}{E}(n+\frac{1}{2})]^{\frac{1}{2}}\frac{\partial f_0}{\partial E}dE$$

In this expression now we introduce the explicit expression of f_0 and find

$$I=-C\frac{\partial}{\partial\zeta}\sum_{n=o}^{\infty}\int_{2\mu H(n+\frac{1}{2})}^{\infty}[1-\frac{2\mu H}{E}(n+\frac{1}{2})]^{\frac{1}{2}}\frac{dE}{1+e^{\frac{E-\zeta}{KT}}}$$

Here now we use the following abbreviations

$$E=2\mu H\varepsilon,\quad \zeta=2\mu H\varepsilon_0\quad \text{and } \theta=\frac{kT}{2\mu H}$$

and get

$$I=-C\frac{\partial}{\partial\varepsilon_0}\sum_{n=o}^{\infty}\int_{n+\frac{1}{2}}^{\infty}(1-\frac{n+\frac{1}{2}}{\varepsilon})^{\frac{1}{2}}\frac{d\varepsilon}{1+e^{\frac{E-\zeta}{RT}}}$$

$$=-C\frac{\partial}{\partial\varepsilon_0}\sum_{n=o}^{\infty}\{\frac{2}{3}\frac{(\varepsilon-n-\frac{1}{2})^{\frac{3}{2}}}{\varepsilon^{\frac{1}{2}}(1+e^{\frac{\varepsilon-\varepsilon0}{\theta}})}\Big]_{n+\frac{1}{2}}^{\infty}+\frac{2}{3}\int_{n+\frac{1}{2}}^{\infty}(\varepsilon-n-\frac{1}{2})^{\frac{3}{2}}(\frac{1}{2}\frac{\varepsilon^{-\frac{1}{2}}}{1+e^{\frac{\varepsilon-\varepsilon0}{\theta}}}$$

$$+\frac{1}{\theta}\frac{\varepsilon^{-\frac{1}{2}}}{(1+e^{\frac{\varepsilon-\varepsilon0}{\theta}})(1+e^{-\frac{\varepsilon-\varepsilon0}{\theta}})})d\varepsilon$$

Because of the limits the first term vanishes and the remaining integrals are now computed with the Poisson-sum formula.

$$\sum_{n=o}^{\infty}\varphi(n+\frac{1}{2})=\sum_{n=-\infty}^{\infty}(-)^n\int_{o}^{\infty}e^{-2\pi int}\varphi(t)dt$$

with this we get then

$$I=-C\frac{\partial}{\partial\varepsilon_0}\frac{2}{3}\sum_{n=-\infty}^{\infty}(-)^n\int_{o}^{\infty}e^{-2\pi int}dt\int_{t}^{\infty}(\varepsilon-t)^{\frac{3}{2}}[\frac{1}{2}\frac{\varepsilon^{-\frac{3}{2}}}{1+e^{\frac{\varepsilon-\varepsilon0}{\theta}}}+\frac{\varepsilon^{-\frac{1}{2}}}{\theta(1+e^{\frac{\varepsilon-\varepsilon0}{\theta}})(1+e^{-\frac{\varepsilon-\varepsilon0}{\theta}})}]d\varepsilon$$

Here we exchange the order of integrations and find

$$I=-C\frac{\partial}{\partial\varepsilon_0}\frac{2}{3}\sum_{n=-\infty}^{\infty}(-)^n\int_{o}^{\infty}[\frac{\frac{1}{2}\varepsilon^{-\frac{3}{2}}}{1+e^{\frac{\varepsilon-\varepsilon0}{\theta}}}+\frac{1}{\theta}\frac{\varepsilon^{-\frac{1}{2}}}{(1+e^{\frac{\varepsilon-\varepsilon0}{\theta}})(1+e^{-\frac{\varepsilon-\varepsilon0}{\theta}})}]d\varepsilon\times\int_{o}^{\varepsilon}(\varepsilon-t)^{\frac{3}{2}}e^{-2\pi int}dt$$

Here we compute now

$$\varphi_n=\int_{0}^{\varepsilon}(\varepsilon-t)^{\frac{3}{2}}e^{-2\pi int}dt=\int_{o}^{\varepsilon}x^{\frac{3}{2}}e^{-2\pi in(\varepsilon-x)}dx=e^{-2\pi in\varepsilon}[\frac{x^{\frac{3}{2}}}{2\pi in}e^{2\pi inx}\Big|_{o}^{\varepsilon}-\frac{3}{2}\int_{o}^{\varepsilon}\frac{x^{\frac{1}{2}}e^{2\pi inx}}{2\pi in}dx]$$

$$=\frac{\varepsilon^{\frac{3}{2}}}{2\pi in}+\frac{3}{2}\frac{\varepsilon^{\frac{1}{2}}}{(2\pi n)^2}-\frac{3}{4}\frac{e^{-2\pi in\varepsilon}}{(2\pi n)^2}\int_{o}^{\varepsilon}x^{-\frac{1}{2}}e^{2\pi inx}dx$$

Because we need the expression mostly for large values of ε we replace the upper limit in the last integral by infinite and get

$$\varphi_n = \frac{\varepsilon^{\frac{3}{2}}}{2\pi i n} + \frac{3}{2}\frac{\varepsilon^{\frac{1}{2}}}{(2\pi n)^2} - \frac{3}{4}\frac{e^{-2\pi i n\varepsilon}}{(2\pi n)^2}\pi^{\frac{1}{2}} e^{i\frac{\pi}{4}}$$

This is the value of the integral, when n is differnt from zero. For n=0 we get

$$\varphi_0 = \int_0^{\varepsilon}(\varepsilon-t)^{\frac{3}{2}}dt = \frac{2}{5}\varepsilon^{\frac{5}{2}}$$

Substituting these values in the expression for the current and taking together the terms with positive and negative n we obtain.

$$I = -C\frac{\partial}{\partial\varepsilon_0}\frac{2}{3}\int_0^{\infty}\Big[\frac{\frac{1}{2}\varepsilon^{-\frac{3}{2}}}{1+e^{\frac{\varepsilon-\varepsilon_0}{\theta}}} + \frac{\frac{1}{\theta}\varepsilon^{-\frac{1}{2}}}{(1+e^{\frac{\varepsilon-\varepsilon_0}{\theta}})(1+e^{-\frac{\varepsilon-\varepsilon_0}{\theta}})}\Big]$$

$$\times\Big[\Big(\frac{2}{5}\varepsilon^{\frac{5}{2}} + \sum_{n=1}^{\infty}(-)^n\Big(\frac{3\varepsilon^{\frac{1}{2}}}{(2\pi n)^2} - \frac{3\sqrt{\pi}}{2}\frac{\cos(2\pi n\varepsilon-\frac{\pi}{4})}{(2\pi n)^{\frac{5}{2}}}\Big)\Big]d\varepsilon$$

Here we carry out the differentiation with respect to ε_0 in the first term and get.

$$I = -C\frac{2}{3\theta}\int_0^{\infty}\Big[\frac{1}{2}\frac{\varepsilon^{-\frac{3}{2}}}{(1+e^{\frac{\varepsilon-\varepsilon_0}{\theta}})(1+e^{-\frac{\varepsilon-\varepsilon_0}{\theta}})} + \frac{\partial}{\partial\varepsilon_0}\frac{\varepsilon^{-\frac{1}{2}}}{(1+e^{\frac{\varepsilon-\varepsilon_0}{\theta}})(1+e^{\frac{\varepsilon-\varepsilon_0}{\theta}})}\Big]$$

$$\times\Big[\frac{2}{5}\varepsilon^{\frac{5}{2}} - \frac{1}{16}\varepsilon^{\frac{1}{2}} - \frac{3\sqrt{\pi}}{2}\sum_{n=1}^{\infty}(-)^n\frac{\cos(2\pi n\varepsilon-\frac{\pi}{4})}{(2\pi n)^{\frac{5}{2}}}\Big]d\varepsilon$$

Here now we introduce the assumption that ε_0/θ is a very large number $\frac{\varepsilon_0}{\theta} = \frac{\zeta}{RT} \gg 1$

Then the integrals can easily be computed

$$\int_0^{\infty}\frac{\varepsilon^n\, d\varepsilon}{(1+e^{\frac{\varepsilon-\varepsilon_0}{\theta}})(1+e^{-\frac{\varepsilon-\varepsilon_0}{\theta}})} = \theta\varepsilon_0^{\,n}$$

For the computation of the integral

$$\int_0^{\infty}\frac{\cos(2\pi n\varepsilon-\frac{\pi}{4})d\varepsilon}{(1+R^{\frac{\varepsilon-\varepsilon_0}{\theta}})(1+R^{-\frac{\varepsilon-\varepsilon_0}{\theta}})} = \theta\int_{-\infty}^{\infty}\frac{\cos[2\pi n(\varepsilon_0+\theta x)-\frac{\pi}{4}]}{(1+e^x)(1+e^{-x})}dx$$

we consider

$$\int_{-\infty}^{\infty}\frac{e^{i[2\pi n(\varepsilon_0+\theta x)-\frac{\pi}{4}]}}{(1+e^x)(1+e^{-x})}dx = e^{i(2\pi n\varepsilon_0-\frac{\pi}{4})}\int_{-\infty}^{\infty}\frac{e^{2\pi i n\theta x}}{(1+e^x)(1+e^{-x})}dx.$$

$$\int_{-\infty}^{\infty}\frac{e^{i\alpha x}dx}{(R+e^x)(1+e^{-x})} = \int_{-\infty}^{\infty}\frac{\cos\alpha x dx}{(1+e^x)(1+e^{-x})} = 2\int_0^{\infty}\frac{\cos\alpha x dx}{(1+e^x)(1+e^{-x})}$$

$$= \Big[-2\frac{\cos\alpha x}{1+e^x}\Big]_0^{\infty} - 2\alpha\int_0^{\infty}\frac{\sin\alpha x}{1+R^x}dx = 1 + 2\alpha^2\sum_{v=1}^{\infty}\frac{(-)^v}{v^2+\alpha^2} = \frac{\pi\alpha}{\sinh\pi\alpha}$$

Thus we obtain our integral as the real part of the considered one.

$$\int_0^\infty \frac{\cos(2\pi n\varepsilon-\frac{\pi}{4})d\varepsilon}{(1+e^{\frac{\varepsilon-\varepsilon_0}{\theta}})(1+e^{-\frac{\varepsilon-\varepsilon_0}{\theta}})}=\frac{2\pi^2 n\theta^2}{\mathrm{Sin\,h}\,2\pi^2 n\theta}\cos(2\pi n\varepsilon_0-\frac{\pi}{4})$$

With this the electric current takes the form

$$I=-C[\frac{2}{3}\varepsilon_0-\frac{1}{48}\varepsilon_0^{-1}+\sum_{n=1}^{\infty}(-)^n\frac{\pi^{-\frac{3}{2}}\theta\varepsilon_0^{-\frac{1}{2}}}{(2\pi n)^{\frac{1}{2}}}\frac{\sin(2\pi n\ \varepsilon_0-\frac{\pi}{4})}{\mathrm{Sin\,h}\ 2\pi^2 n\theta}]$$

$$=\frac{16\pi e^2 mFl\zeta}{3h^3}[1-\frac{1}{8}(\frac{\mu H}{\zeta})^2+\frac{3\pi}{2}\frac{kT}{\zeta}(\frac{\mu H}{\zeta})^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{(-)^n}{n^{\frac{1}{2}}}\frac{\sin(\frac{n\pi\zeta}{\mu H}-\frac{\pi}{4})}{\sinh\frac{n\pi^2 kT}{\mu H}}]$$

We use the abbreviations

$I_0'=\frac{16\pi e^2 mFl\zeta}{3h^3}$ and I_0 for the current in vanishing magnetic field.

The Fermienergy ζ can be expressed in terms of the Fermienenergy ζ_0 in vanishing mangetic field in the following way.

$$\zeta=\zeta_0\{1+\frac{2}{3}[\frac{1}{8}(\frac{\mu H}{\zeta_0})^2-\sum_{n=1}^{\infty}(-)^n\frac{3\pi}{2n^{\frac{1}{2}}}\frac{kT}{\zeta_0}(\frac{\mu H}{\zeta_0})^{\frac{1}{2}}\frac{\mathrm{Sin}(\frac{n\pi\zeta_0}{\mu H}-\frac{\pi}{4})}{\mathrm{Sinh}\frac{n\pi^2 kT}{\mu H}}]\}$$

With this we obtain

$$I_0'=\frac{16\pi e^2 mFl\zeta_0}{3h^3}\{1+\frac{2}{3}[\frac{1}{8}(\frac{\mu H}{\zeta_0})^2-\frac{3\pi}{2}\frac{kT}{\zeta_0}(\frac{\mu H}{\zeta_0})^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{(-)^n}{n^{\frac{1}{2}}}\frac{\sin(\frac{n\pi\zeta_0}{\mu H}-\frac{\pi}{4})}{\mathrm{Sinh}\frac{n\pi^2 kT}{\mu H}}]\}$$

Here l is the mean free path corresponding to the Fermienergy ζ. We compute it here in terms of the mean free path l_0 corresponding to ζ_0. We assume that l depends on the energy only and get

$$l=l_0+(\zeta-\zeta_0)(\frac{\partial l}{\partial\zeta})_0$$

$$=l_0[1+(\zeta-\zeta_0)(\frac{\partial \ln l}{\partial\zeta})_0]$$

Here we introduce the abbreviation

$$\Lambda=1+\zeta_0(\frac{\partial \ln l}{\partial\zeta})_0$$

$$l=l_0(1+\frac{\zeta-\zeta_0}{\zeta_0}\zeta_0(\frac{\partial \ln l}{\partial\zeta})_0=l_0\left(1+(\Lambda-1)\frac{\zeta-\zeta_0}{\zeta_0}\right)$$

We find then with

$$I_0=\frac{16\pi e^2 mFl_0\zeta_0}{3h^3}$$

$$I'_0=I_0\{1+\frac{2A}{3}[\frac{1}{8}(\frac{\mu H}{\zeta_0})^2-\frac{kT}{\zeta_0}(\frac{\mu H}{\zeta_0})^{\frac{1}{2}}\sum_{n=1}^{\infty}(-)^n\frac{3\pi}{2n^{\frac{1}{2}}}\frac{\sin(\frac{n\pi\zeta_0}{kT}-\frac{\pi}{4})}{\mathrm{Sinh}\frac{n\pi^2 kT}{\mu H}}]\}$$

We thus obtain

$$\frac{I}{I_0}=\frac{I}{I_0'}\frac{I_0'}{I_0}=1+(\frac{2A}{3}-1)[\frac{1}{8}(\frac{\mu H}{\xi_0})^2-\frac{3\pi}{2}\frac{kT}{\zeta_0}(\frac{\pi H}{\zeta_0})^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{(-)^n}{n^{\frac{1}{2}}}\frac{\sin(\frac{n\pi\zeta}{\mu tl}-\frac{\pi}{4})}{\mathrm{Sinh}\frac{n\pi^2 kT}{\mu H}}]$$

Therefore the resistance ρ in the magnetic field and that ρ_0 without magnetic field stay in the relation

$$\frac{I}{I_0}=\frac{\rho_0}{\rho}$$

Thus we obtain for $\triangle\rho=\rho-\rho_0$ the relation

$$\frac{I}{I_0}-1=\frac{\rho_0}{\rho}-1=\frac{\rho_0-\rho}{\rho}=-\frac{\triangle\rho}{\rho}$$

$$\frac{\Delta\rho}{\rho}=(\frac{1}{8}-\frac{A}{12})(\frac{\mu H}{\zeta})^2-(1-\frac{2A}{3})\frac{3\pi}{2}\frac{kT}{\zeta}(\frac{\mu H}{\zeta})^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{(-)^n}{n^{\frac{1}{2}}}\frac{\sin(\frac{n\pi\zeta}{\mu H}-\frac{\pi}{4})}{\mathrm{Sinh}\frac{n\pi^2 kT}{\mu H}}$$

Here the first term is that obtained by Sommerfeld and Bartlett. The second term gives a periodic variation superimposed upon the first one. Such variations have been found in experiment for strong magnetic fields and low temperatures. On account of the many approximations made in our computatin, it is doubtful, if our result gives a satisfactory representation of the experimental results. However it is interesting that such a simple model gives these variations.

(1) A. Sommrfeld und und B.W. Bartlett Phy. Z.36, 894, (1935)

根據實驗報告，磁阻因磁場而行週期變化。爲解釋此種性質，今特出一簡單模型，並加以計算。其結論：磁阻之此種週期變化係起因電子軌跡在磁場中之量子化。 —克流—